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DYNAMICS OF AN AXIALLY EXTENDING AND ROTATING CANTILEVER BEAM INCLUDING THE EFFECT OF GRAVITY

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Abstract—A system of approximate differential equations in matrix form is derived for the dynamics of an axially extending and rotating beam including the effect of gravity using a Lagrangian approach and the assumed mode method. A simple series of single-term functions is used for the assumed mode in place of the beam functions used in the other reported studies. The transverse motions in the two principal orthogonal directions of the cross-section of the beam are coupled when the setting angle, which is the angle between the axis of rotation and one of the principal axes of the cross section of the beam, is not equal to 0° or 90°. Numerical simulations are performed for various combinations of prescribed axial and rotational motions of the beam.

1. INTRODUCTION

The vibration and dynamics of rotating beams have received significant attention as the beams are often used as simple models for propellers, turbine blades and satellite booms. The natural frequencies of rotating beams, including the effect of tip mass, follower force, root flexibility, stretching and setting angle, are reported in works by Schilhansl (1958), Carnegie (1959), Rubinstein and Stadter (1972), Pnuelli (1972), Kumar (1974), Stafford and Giurgiutiu (1975), Anderson (1975), Wang *et al.* (1976), Putter and Manor (1978), Hoa (1979), Bhat (1986), Abbas (1986), Stephen and Wang (1986), Kim and Dickinson (1987), Bauer and Eidel (1988), and Lee (1993). Kane and Ryan (1987) presented a general formulation based on the method of Kane's dynamics for computing the dynamics of a cantilever beam attached to a moving base with a prescribed motion of rotation and translation. All of these studies are for beams without any axial motion. On the other hand, there are relatively few studies on the dynamics of axially extending beams. The dynamics of an axially extending cantilever beam without rotation was studied by Tabarrok *et al.* (1974). Wang and Wei (1987) analyzed the motion of a slender prismatic beam using Galerkin approximation. Krishnamurthy (1989) presented the dynamic equations of a flexible cylindrical manipulator. In his study, the tail section of the robotic arm is included in the analysis so that the total length of the arm remains fixed. Yuh and Young (1991) presented the dynamic equations for the planar motion of an axially moving beam in rotation. For these studies, the gravitational effect is neglected. Moreover, the beam is located in such a manner that the lateral motions of the beam are either parallel or normal to the axis of rotation. The effect of setting angle, which is the angle between the axis of rotation and one of the principal axes of the cross-section of the beam, has not been investigated. Such a configuration for an axially extending and rotating beam can be found in robotic manipulators and mechanisms with extensible arms.

The present formulation uses the Lagrangian approach and restricts the hub where the beam is attached to a general rotational motion about a fixed axis. The beam is clamped to the hub with an arbitrary setting angle. The effect of gravity is included. The equations of motion are then reduced to the form of a system of matrix equations using the assumed mode method.

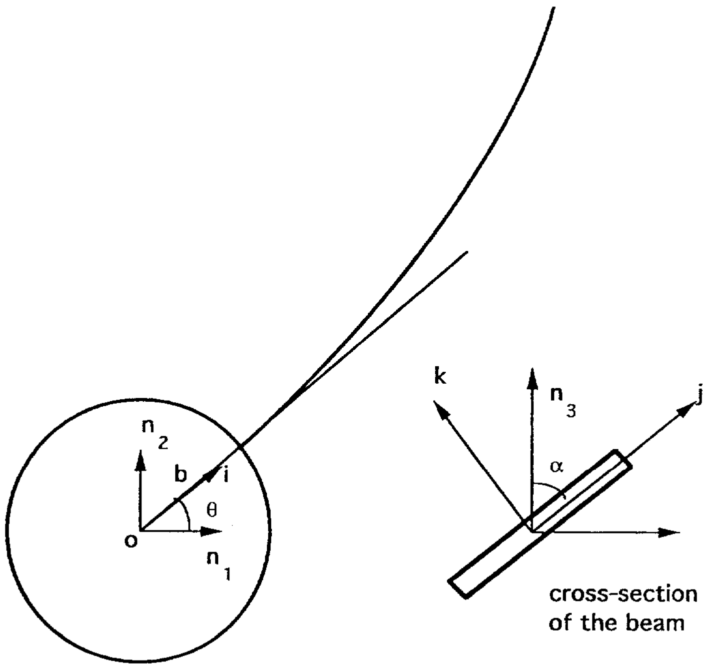

Fig. 1. An axially moving and rotating cantilever beam.

2. THEORY AND FORMULATIONS

The beam considered is assumed to be a uniform beam of length l attached to a hub of radius b, shown in Fig. 1. A set of mutually perpendicular unit vectors, $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ is assumed to be fixed in a Newtonian reference frame N. The unit vector $\mathbf{n}_3$ is made to coincide with the axis of rotation of the hub. A second set of mutually perpendicular unit vectors, **i, j** and **k** is fixed in the undeformed beam with the **i** vector parallel to the undeformed beam. The unit vectors **j** and **k** are made to coincide with the two principal orthogonal directions of the rectangular cross-section of the beam. The angle α between **j** and $\mathbf{n}_3$ is called the setting angle of the beam. The position of the hub is measured by an angle θ, defined as the angle between **i** and $\mathbf{n}_1$ shown in Fig. 1. Flexibility of the beam in the axial direction **i** is assumed to be negligible compared to the two lateral directions **j** and **k**.

The position vector of a general point e on the deformed beam is given by

$$\mathbf{e} = r\mathbf{i} + u_2(r, t)\mathbf{j} + u_3(r, t)\mathbf{k}. \tag{1}$$

The variables u_2 and u_3 are the transverse deflections of the beam in the **j** and **k** directions, respectively. The velocity at the point is

$$\mathbf{v}_e = (U - \dot{\theta}u_2 \sin\alpha + \dot{\theta}u_3 \cos\alpha)\mathbf{i} + \left[(b+r)\dot{\theta}\sin\alpha + \frac{\partial u_2}{\partial t} + U\frac{\partial u_2}{\partial r}\right]\mathbf{j} + \left[-(b+r)\dot{\theta}\cos\alpha + \frac{\partial u_3}{\partial t} + U\frac{\partial u_3}{\partial r}\right]\mathbf{k}, \tag{2}$$

where U is the prescribed axial speed of the beam at the root of the cantilever and $\dot{\theta}$ is defined as $\mathrm{d}\theta/\mathrm{d}t$.

The kinetic energy T of the beam is

$$T = \tfrac{1}{2}m\int_0^{l(t)}\left\{(U-\dot{\theta}u_2\sin\alpha+\dot{\theta}u_3\cos\alpha)^2+\left((b+r)\dot{\theta}\sin\alpha+\frac{\partial u_2}{\partial t}+U\frac{\partial u_2}{\partial r}\right)^2 + \left(-(b+r)\dot{\theta}\cos\alpha+\frac{\partial u_3}{\partial t}+U\frac{\partial u_3}{\partial r}\right)^2\right\}\mathrm{d}r, \quad (3)$$

where m is the mass of the beam per unit length.

The expression for the kinetic energy involves a coupling term of $m\dot{\theta}^2u_2u_3\sin\alpha\cos\alpha$. When α is equal to 0° or 90°, this coupling term between u_2 and u_3 vanishes. For these two cases, one of the principal axes of the cross-section of the beam is parallel to the axis of rotation of the hub. The motion of the beam in a plane normal to the axis of rotation is uncoupled from the out-of-plane motion of the beam.

Assuming Euler's beam theory, the elastic strain energy of the beam due to bending is

$$V_\epsilon = \tfrac{1}{2}\int_0^{l(t)}\left[EI_k\left(\frac{\partial^2 u_2}{\partial r^2}\right)^2+EI_j\left(\frac{\partial^2 u_3}{\partial r^2}\right)^2\right]\mathrm{d}r, \quad (4)$$

where E, I_j and I_k are the Young's modulus and the central principal second moment of area of the cross-section of the beam about the **j** axis and **k** axis, respectively.

The potential energy due to the axial forces, f_r, is

$$V_a = \tfrac{1}{2}\int_0^{l(t)} f_r\left\{\left(\frac{\partial u_2}{\partial r}\right)^2+\left(\frac{\partial u_3}{\partial r}\right)^2\right\}\mathrm{d}r. \quad (5)$$

The axial forces, f_r, in the direction along the beam are

$$f_r = -m\int_r^{l(t)}\left\{\dot{U}-(b+x)\dot{\theta}^2\right\}\mathrm{d}x \quad (6)$$

$$= -m\left\{\dot{U}(l-r)-b\dot{\theta}^2(l-r)-\tfrac{1}{2}\dot{\theta}^2(l^2-r^2)\right\}.$$

Terms involving u_2 and u_3 in f_r have been neglected as the contribution of such terms to V_a will be of the order of u_2^3 or u_3^3.

The work done due to the effect of gravity with $-g\mathbf{n}_3$ defined as the gravitational acceleration having a magnitude of 9.81 m s^{-2}, is given by

$$W = -mg\int_0^{l(t)}(\cos\alpha u_2+\sin\alpha u_3)\,\mathrm{d}r. \quad (7)$$

If the quantities u_2 and u_3 are expressed as

$$u_2 = \sum_{i=1}^{n} q_i(t)\phi_i(r,l) \quad (8)$$

$$u_3 = \sum_{i=1}^{n} p_i(t)\phi_i(r,l), \quad (9)$$

where ϕ_i are spatial functions that satisfy the geometric boundary conditions at the clamped end of the beam. The functions ϕ_i are functions of time since l changes with time. In the reported works by Tabarrok *et al.* (1974), and Yuh and Young (1991), the functions ϕ_i are assumed to be the eigenfunctions of a stationary cantilever beam of length l. The eigenfunctions are

$$\phi_i = \frac{1}{\sqrt{l}}\left[\cos\frac{\varepsilon_i}{l}r - \cosh\frac{\varepsilon_i}{l}r - \alpha_i\left(\sin\frac{\varepsilon_i}{l}r - \sinh\frac{\varepsilon_i}{l}r\right)\right], \tag{10}$$

where ε_i are the roots of the equation

$$1 + \cos\varepsilon_i \cosh\varepsilon_i = 0 \tag{11}$$

and

$$\alpha_i = \frac{\cos\varepsilon_i + \cosh\varepsilon_i}{\sin\varepsilon_i + \sinh\varepsilon_i}. \tag{12}$$

The resulting $\dot{\phi}_i$ is

$$\dot{\phi}_i = -\frac{U}{2l}\phi_i - \frac{Ur}{l}\phi_i', \tag{13}$$

where $\phi_i' = \partial\phi_i/\partial r$.

Tabarrok *et al.* (1974), in their work for axially moving beams without rotation, reported that some difficulties in computation were encountered for high values of ε_i as a result of the relatively large magnitudes of the hyperbolic terms. Moreover, all the matrices except the mass matrix in the final equation of motion in matrix form are not diagonal. There is therefore no significant advantage, besides having a diagonal mass matrix, in using the set of orthogonal beam functions. For the present analysis, ϕ_i are assumed to be

$$\phi_i = \left(\frac{r}{l}\right)^{1+i}. \tag{14}$$

The assumed functions satisfy the zero slope and zero displacement geometric boundary conditions of the clamped end of the beam. The zero moment and zero shear force conditions, which are the natural boundary conditions at the free end, are not satisfied by these functions.

The functions $\dot{\phi}_i$ are given by

$$\dot{\phi}_i = -(1+i)\frac{U}{l}\phi_i. \tag{15}$$

The above expression is simpler in form than the corresponding expression of $\dot{\phi}_i$ using orthogonal beam functions.

The assumed functions for ϕ_i enable the kinetic energy, the strain energy, the potential energy, and the work done by the gravitational forces to be expressed in matrix form. The parts of kinetic energy, potential energy, strain energy, and work done by the gravitational force involving u_2 are

$$\begin{aligned} T_2 = {} & \tfrac{1}{2}m\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{H}\dot{\mathbf{q}} - \tfrac{1}{2}m\frac{U}{l}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{E}_1\mathbf{q} - \tfrac{1}{2}m\frac{U}{l}\mathbf{q}^{\mathrm{T}}\mathbf{E}_2\dot{\mathbf{q}} + \tfrac{1}{2}m\frac{U^2}{l^2}\mathbf{q}^{\mathrm{T}}\mathbf{F}\mathbf{q} \\ & + m\dot{\theta}\sin\alpha\mathbf{B}_1{}^{\mathrm{T}}\dot{\mathbf{q}} - m\dot{\theta}\sin\alpha\frac{U}{l}\mathbf{B}_2{}^{\mathrm{T}}\mathbf{q} + mU\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{Y}_1\mathbf{q} - m\frac{U^2}{l}\mathbf{q}^{\mathrm{T}}\mathbf{Y}_2\mathbf{q} \\ & + \tfrac{1}{2}m\dot{\theta}^2\sin^2\alpha\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{q} - mU\dot{\theta}\sin\alpha\mathbf{\Phi}^{\mathrm{T}}\mathbf{q} - m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{q}^{\mathrm{T}}\mathbf{H}\mathbf{p} \\ & + \tfrac{1}{2}mU^2\mathbf{q}^{\mathrm{T}}\mathbf{Q}\mathbf{q} + m\dot{\theta}U\sin\alpha\mathbf{R}^{\mathrm{T}}\mathbf{q} \end{aligned} \tag{16}$$

$$V_{e2} = \tfrac{1}{2} EI_k \mathbf{q}^{\mathrm{T}}\mathbf{M}\mathbf{q} \tag{17}$$

$$V_a 2 = \tfrac{1}{2} m(b\dot{\theta}^2 - \dot{U})\mathbf{q}^{\mathrm{T}}\mathbf{S}_1\mathbf{q} + \tfrac{1}{2} m\dot{\theta}^2\mathbf{q}^{\mathrm{T}}\mathbf{S}_2\mathbf{q} \tag{18}$$

$$W_2 = -mg\cos\alpha\mathbf{\Phi}^{\mathrm{T}}\mathbf{q}, \tag{19}$$

where $\mathbf{H}$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{F}$, $\mathbf{Q}$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{M}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are matrices defined as

$$(\mathbf{H})_{ij} = \int_0^l \phi_i \phi_j \,\mathrm{d}r = \frac{l}{3+i+j} \tag{20}$$

$$(\mathbf{E}_1)_{ij} = (1+j)(\mathbf{H})_{ij} \tag{21}$$

$$(\mathbf{E}_2)_{ij} = (1+i)(\mathbf{H})_{ij} \tag{22}$$

$$(\mathbf{F})_{ij} = (1+i)(1+j)(\mathbf{H})_{ij} \tag{23}$$

$$(\mathbf{Q})_{ij} = \int_0^l \phi_i'\phi_j' \,\mathrm{d}r = \frac{(1+i)(1+j)}{(1+i+j)l} \tag{24}$$

$$(\mathbf{Y}_1)_{ij} = \int_0^l \phi_i\phi_j' \,\mathrm{d}r = \frac{1+j}{2+i+j} \tag{25}$$

$$(\mathbf{Y}_2)_{ij} = (1+i)(\mathbf{Y}_1)_{ij} \tag{26}$$

$$(\mathbf{M})_{ij} = \int_0^l \phi_i''\phi_j'' \mathrm{d}r = \frac{(1+i)(1+j)ij}{(i+j-1)l^3} \tag{27}$$

$$(\mathbf{S}_1)_{ij} = \int_0^l (1-r)\phi_i'\phi_j' \,\mathrm{d}r = \frac{(1+i)(1+j)}{(1+i+j)(2+i+j)} \tag{28}$$

$$(\mathbf{S}_2)_{ij} = \int_0^l \tfrac{1}{2}(l^2-r^2)\phi_i'\phi_j' \,\mathrm{d}r = \frac{(1+i)(1+j)l}{(1+i+j)(3+i+j)}. \tag{29}$$

It can be seen from the above expressions that $\mathbf{H}$, $\mathbf{F}$, $\mathbf{Q}$, $\mathbf{M}$, $\mathbf{S}_1$ and $\mathbf{S}_2$ are symmetric matrices. Moreover, some of these matrices are functions of l and therefore time-dependent. The functions ϕ_i' and ϕ_i'' denote the first and second derivatives of ϕ_i with respect to r.

The vectors $\mathbf{q}$, $\mathbf{p}$, $\dot{\mathbf{q}}$, and $\dot{\mathbf{p}}$ are $n \times 1$ column vectors consisting of q_i, p_i, $\dot{q}_i$ and $\dot{p}_i$. The vectors $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{R}$ and $\mathbf{\Phi}$ are $n \times 1$ column vectors defined by

$$(\mathbf{B}_1)_i = \int_0^l (r+b)\phi_i \,\mathrm{d}r = \frac{l^2}{3+i} + b(\mathbf{\Phi})_i \tag{30}$$

$$(\mathbf{B}_2)_i = (1+i)(\mathbf{B}_1)_i \tag{31}$$

$$(\mathbf{R})_i = \int_0^l (r+b)\phi_i' \,\mathrm{d}r = \frac{(1+i)l}{2+i} + b \tag{32}$$

$$(\mathbf{\Phi})_i = \int_0^l \phi_i \,\mathrm{d}r = \frac{l}{2+i}. \tag{33}$$

Similarly, the kinetic energy, potential energy, strain energy, and work done by the gravitational forces involving u_3 are

$$T_3 = \tfrac{1}{2}m\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{H}\dot{\mathbf{p}} - \tfrac{1}{2}m\frac{U}{l}\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{E}_1\mathbf{p} - \tfrac{1}{2}m\frac{U}{l}\mathbf{p}^{\mathrm{T}}\mathbf{E}_2\dot{\mathbf{p}} + \tfrac{1}{2}m\frac{U^2}{l^2}\mathbf{p}^{\mathrm{T}}\mathbf{F}\mathbf{p} - m\dot{\theta}\cos\alpha\mathbf{B}_1^{\mathrm{T}}\dot{\mathbf{p}}$$
$$+ m\dot{\theta}\cos\alpha\frac{U}{l}\mathbf{B}_2^{\mathrm{T}}\mathbf{p} + mU\dot{\mathbf{p}}^{\mathrm{T}}\mathbf{Y}_1\mathbf{p} - m\frac{U^2}{l}\mathbf{p}^{\mathrm{T}}\mathbf{Y}_2\mathbf{p} + \tfrac{1}{2}m\dot{\theta}^2\cos^2\alpha\mathbf{p}^{\mathrm{T}}\mathbf{H}\mathbf{p} + mU\dot{\theta}\cos\alpha\mathbf{\Phi}^{\mathrm{T}}\mathbf{p}$$
$$- m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{p}^{\mathrm{T}}\mathbf{H}\mathbf{q} + \tfrac{1}{2}mU^2\mathbf{p}^{\mathrm{T}}\mathbf{Q}\mathbf{p} - m\dot{\theta}U\cos\alpha\mathbf{R}^{\mathrm{T}}\mathbf{p} \tag{34}$$

$$V_{\varepsilon 3} = \tfrac{1}{2}EI_j\mathbf{p}^{\mathrm{T}}\mathbf{M}\mathbf{p} \tag{35}$$

$$V_{a3} = \tfrac{1}{2}m(b\dot{\theta}^2 - \dot{U})\mathbf{p}^{\mathrm{T}}\mathbf{S}_1\mathbf{p} + \tfrac{1}{2}m\dot{\theta}^2\mathbf{p}^{\mathrm{T}}\mathbf{S}_2\mathbf{p} \tag{36}$$

$$W_3 = -mg\sin\alpha\mathbf{\Phi}^{\mathrm{T}}\mathbf{p}. \tag{37}$$

The term $m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{p}^{\mathrm{T}}\mathbf{H}\mathbf{q}$ is the common coupling term which appears in both T_2 and T_3. This term vanishes when $\alpha = 0^\circ$ or 90°.

The Lagrangians of the beam involving u_2 and u_3 can be expressed separately as

$$L_2 = T_2 - V_{\varepsilon 2} - V_{a2} + W_2 \tag{38}$$

$$L_3 = T_3 - V_{\varepsilon 3} - V_{a3} + W_3. \tag{39}$$

The application of Hamilton's principle for a cantilever beam with variable length was presented by Tabarrok *et al.* (1974). Using the same principle for the above Lagrangians of the beam, bearing in mind that both U and l are functions of t, the Euler–Lagrange equations are

$$m\mathbf{H}\ddot{\mathbf{q}} + \left\{mU(\mathbf{Y}_1 - \mathbf{Y}_1^{\mathrm{T}}) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_1^{\mathrm{T}} - \mathbf{E}_1) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_2 - \mathbf{E}_2^{\mathrm{T}}) + m\frac{U}{l}\mathbf{H}\right\}\dot{\mathbf{q}}$$
$$+ \left\{m\frac{U^2}{l}(\mathbf{Y}_2 + \mathbf{Y}_2^{\mathrm{T}}) - \tfrac{1}{2}m\frac{\dot{U}}{l}(\mathbf{E}_1 + \mathbf{E}_2^{\mathrm{T}}) + m\dot{U}\mathbf{Y}_1 - m\frac{U^2}{l^2}\mathbf{F}\right.$$
$$\left. - m\dot{\theta}^2\sin^2\alpha\mathbf{H} - mU^2\mathbf{Q} + EI_k\mathbf{M} + m(b\dot{\theta}^2 - \dot{U})\mathbf{S}_1 + m\dot{\theta}^2\mathbf{S}_2\right\}\mathbf{q} + mg\cos\alpha\mathbf{\Phi}$$
$$+ m\ddot{\theta}\sin\alpha\mathbf{B}_1 + m\frac{U}{l}\dot{\theta}\sin\alpha\mathbf{B}_2 + mU\dot{\theta}\sin\alpha\mathbf{\Phi} - mU\dot{\theta}\sin\alpha\mathbf{R} + m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{H}\mathbf{p} = 0 \tag{40}$$

$$m\mathbf{H}\ddot{\mathbf{p}} + \left\{mU(\mathbf{Y}_1 - \mathbf{Y}_1^{\mathrm{T}}) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_1^{\mathrm{T}} - \mathbf{E}_1) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_2 - \mathbf{E}_2^{\mathrm{T}}) + m\frac{U}{l}\mathbf{H}\right\}\dot{\mathbf{p}}$$
$$+ \left\{m\frac{U^2}{l}(\mathbf{Y}_2 + \mathbf{Y}_2^{\mathrm{T}}) - \tfrac{1}{2}m\frac{\dot{U}}{l}(\mathbf{E}_1 + \mathbf{E}_2^{\mathrm{T}}) + m\dot{U}\mathbf{Y}_1 - m\frac{U^2}{l^2}\mathbf{F}\right.$$
$$\left. - m\dot{\theta}^2\cos^2\alpha\mathbf{H} - mU^2\mathbf{Q} + EI_j\mathbf{M} + m(b\dot{\theta}^2 - \dot{U})\mathbf{S}_1 + m\dot{\theta}^2\mathbf{S}_2\right\}\mathbf{p} + mg\sin\alpha\mathbf{\Phi}$$

$$-m\ddot{\theta}\cos\alpha\mathbf{B}_1 - m\frac{U}{l}\dot{\theta}\cos\alpha\mathbf{B}_2 - mU\dot{\theta}\cos\alpha\mathbf{\Phi} + mU\dot{\theta}\cos\alpha\mathbf{R} + m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{Hq} = 0. \tag{41}$$

In deriving the above equations of motion, one should bear in mind that some of the matrices are functions of l and therefore time-dependent. For example, the time derivative of $m\mathbf{H}\dot{\mathbf{p}}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{H}\dot{\mathbf{p}}) = m\mathbf{H}\ddot{\mathbf{p}} + m\dot{\mathbf{H}}\dot{\mathbf{p}} \tag{42}$$

$$= m\mathbf{H}\ddot{\mathbf{p}} + m\frac{U}{l}\mathbf{H}\dot{\mathbf{p}}$$

as $\mathrm{d}l/\mathrm{d}t = U$. This is due to the assumption that flexibility in the axial direction is assumed to be negligible compared to the two lateral directions of the beam.

If there is no axial motion, $U = \dot{U} = 0$. The equations of motion can be simplified as

$$m\mathbf{H}\ddot{\mathbf{q}} + \{EI_k\mathbf{M} + mb\dot{\theta}^2\mathbf{S}_1 + m\dot{\theta}^2\mathbf{S}_2 - m\dot{\theta}^2\sin^2\alpha\mathbf{H}\}\mathbf{q} + m\ddot{\theta}\sin\alpha\mathbf{B}_1$$
$$+ m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{Hp} + mg\cos\alpha\mathbf{\Phi} = 0 \tag{43}$$

$$m\mathbf{H}\ddot{\mathbf{p}} + \{EI_j\mathbf{M} + mb\dot{\theta}^2\mathbf{S}_1 + m\dot{\theta}^2\mathbf{S}_2 - m\dot{\theta}^2\cos^2\alpha\mathbf{H}\}\mathbf{p}$$
$$- m\ddot{\theta}\cos\alpha\mathbf{B}_1 + m\dot{\theta}^2\sin\alpha\cos\alpha\mathbf{Hq} + mg\sin\alpha\mathbf{\Phi} = 0. \tag{44}$$

If there is no rotational motion, $\dot{\theta} = \ddot{\theta} = 0$. The equations of motion for an axially moving cantilever beam are

$$m\mathbf{H}\ddot{\mathbf{q}} + \left\{mU(\mathbf{Y}_1 - \mathbf{Y}_1^{\mathrm{T}}) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_1^{\mathrm{T}} - \mathbf{E}_1) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_2 - \mathbf{E}_2^{\mathrm{T}}) + m\frac{U}{l}\mathbf{H}\right\}\dot{\mathbf{q}}$$
$$+ \left\{m\frac{U^2}{l}(\mathbf{Y}_2 + \mathbf{Y}_2^{\mathrm{T}}) - \tfrac{1}{2}m\frac{\dot{U}}{l}(\mathbf{E}_1 + \mathbf{E}_2^{\mathrm{T}}) + m\dot{U}\mathbf{Y}_1 - m\frac{U^2}{l^2}\mathbf{F}\right.$$
$$\left. - mU^2\mathbf{Q} + EI_k\mathbf{M} - m\dot{U}\mathbf{S}_1\right\}\mathbf{q} + mg\cos\alpha\mathbf{\Phi} = 0 \tag{45}$$

$$m\mathbf{H}\ddot{\mathbf{p}} + \left\{mU(\mathbf{Y}_1 - \mathbf{Y}_1^{\mathrm{T}}) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_1^{\mathrm{T}} - \mathbf{E}_1) + \tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_2 - \mathbf{E}_2^{\mathrm{T}}) + m\frac{U}{l}\mathbf{H}\right\}\dot{\mathbf{p}}$$
$$+ \left\{m\frac{U^2}{l}(\mathbf{Y}_2 + \mathbf{Y}_2^{\mathrm{T}}) - \tfrac{1}{2}m\frac{\dot{U}}{l}(\mathbf{E}_1 + \mathbf{E}_2^{\mathrm{T}}) + m\dot{U}\mathbf{Y}_1 - m\frac{U^2}{l^2}\mathbf{F}\right.$$
$$\left. - mU^2\mathbf{Q} + EI_j\mathbf{M} - m\dot{U}\mathbf{S}_1\right\}\mathbf{p} + mg\sin\alpha\mathbf{\Phi} = 0. \tag{46}$$

The two equations are uncoupled without the rotational motion of the hub.

3. RESULTS AND SIMULATIONS

The following functional forms for $l(t)$ and $\theta(t)$ are used for the following numerical simulations:

Table 1. Coefficients in eqns (47) and (48) for various beam motions

Case number	c_1(cm)	c_2(cm)	d_1(rad)	d_2(rad)	T_d(s)
1	35	70	0	0	1.2
2	35	70	0	1.57	1.2
3	105	−70	0	0	1.2
4	105	−70	0	1.57	1.2
5	35	70	0	0	0.2
6	35	70	0	0.1	0.2
7	105	−70	0	0	0.2
8	105	−70	0	0.1	0.2

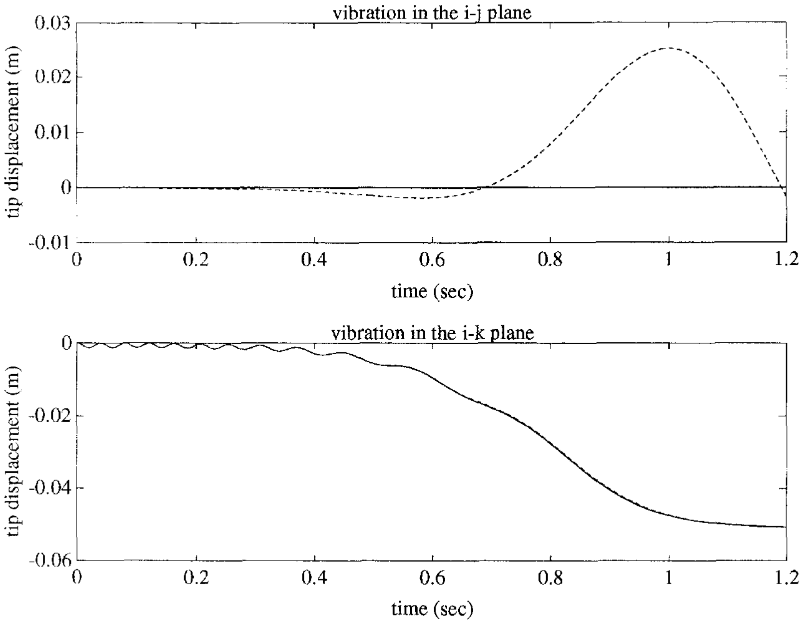

Fig. 2. Tip displacement for the beam with $\alpha = 90^\circ$ for case 1 (——), case 2 (- - -).

$$l(t) = c_1 + \frac{c_2}{T_d}\left[t - \frac{T_d}{2\pi}\sin\frac{2\pi t}{T_d}\right]\text{cm} \tag{47}$$

$$\theta(t) = d_1 + \frac{d_2}{T_d}\left[t - \frac{T_d}{2\pi}\sin\frac{2\pi t}{T_d}\right]\text{cm}. \tag{48}$$

The eight different cases for the simulations are appended in Table 1. The duration of the prescribed motions for cases 1–4 is six times the duration of similar prescribed motions for cases 5–8. The beam is assumed to be a brass rod of diameter 6.35 mm with parameters given by: $EI = 7.981138$ Nm^2, and $m = 0.269$ kg m^{-1}. The radius of the hub, b, is also assumed to be zero. The beam is assumed to be initially straight and stationary. A convergence study for n, the number of terms in the assumed series for u_2 and u_3, is first performed for some of the cases. Numerical values of the tip displacement of the beam for $n = 4$ and $n = 5$ are found to be almost identical. Consequently, n is set to be 4 for the following numerical simulations.

Numerical results are presented in Figs 2–5 for all the eight cases in Table 1 for setting angle $\alpha = 90^\circ$. For this setting angle of the beam, there is no coupling term between $\mathbf{p}$ and $\mathbf{q}$. The two equations of motion (40) and (41) can be reduced to

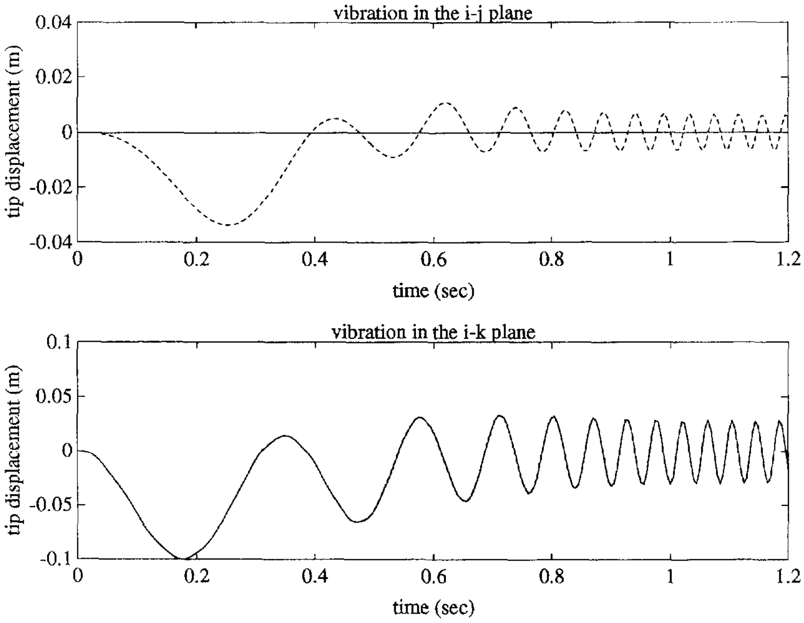

Fig. 3. Tip displacement for the beam with $\alpha = 90^\circ$ for case 3 (——), case 4 (- - -).

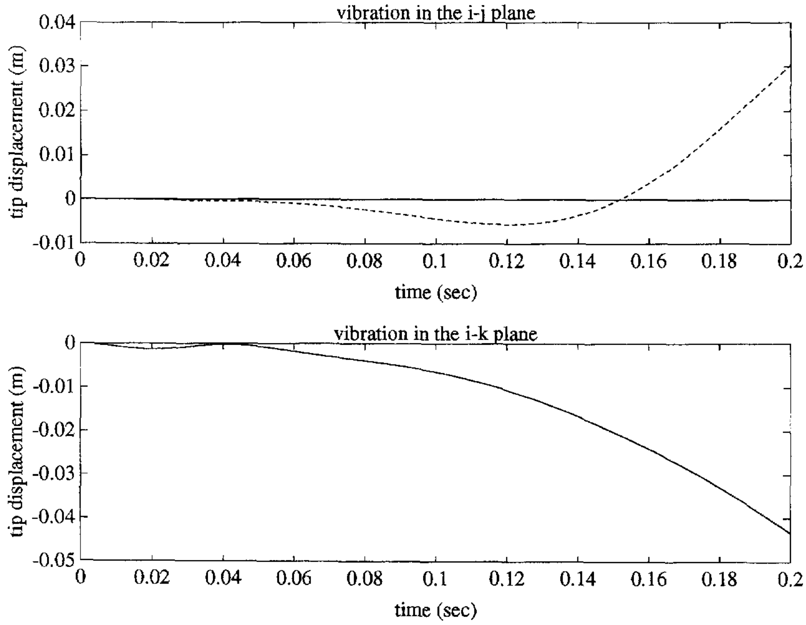

Fig. 4. Tip displacement for the beam with $\alpha = 90^\circ$ for case 5 (——), case 6 (- - -).

$$m\mathbf{H}\ddot{\mathbf{q}} + \left\{ mU(\mathbf{Y}_1 - \mathbf{Y}_1^{\mathrm{T}}) + \tfrac{1}{2} m \frac{U}{l}(\mathbf{E}_1^{\mathrm{T}} - \mathbf{E}_1) + \tfrac{1}{2} m\frac{U}{l}(\mathbf{E}_2-\mathbf{E}_2^{\mathrm{T}}) + m\frac{U}{l}\mathbf{H}\right\}\dot{\mathbf{q}}$$

$$+ \left\{ m\frac{U^2}{l}(\mathbf{Y}_2+\mathbf{Y}_2^{\mathrm{T}}) - \tfrac{1}{2} m \frac{\dot{U}}{l}(\mathbf{E}_1+\mathbf{E}_2^{\mathrm{T}}) + m\dot{U} \mathbf{Y}_1 - m\frac{U^2}{l^2}\mathbf{F}\right.$$

$$\left. - m\dot{\theta}^2\mathbf{H} - mU^2\mathbf{Q} + EI_k\mathbf{M} + m(b\dot{\theta}^2 - \dot{U})\mathbf{S}_1 + m\dot{\theta}^2\mathbf{S}_2\right\}\mathbf{q}$$

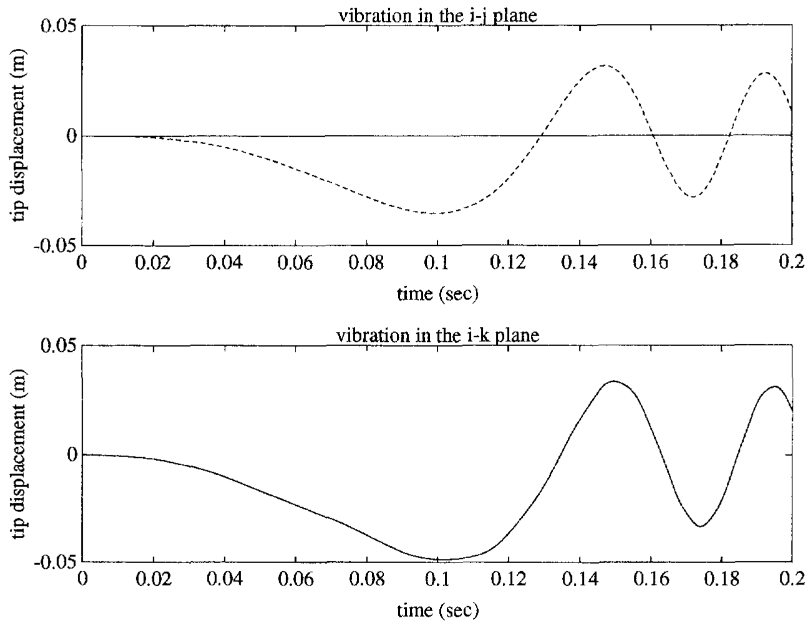

Fig. 5. Tip displacement for the beam with $\alpha = 90^\circ$ for case 7 (——), case 8 (- - -).

$$+m\ddot{\theta}\mathbf{B}_1 + m\frac{U}{l}\dot{\theta}\mathbf{B}_2 + mU\dot{\theta}\mathbf{\Phi} - mU\dot{\theta}\mathbf{R} = 0 \tag{49}$$

$$m\mathbf{H}\ddot{\mathbf{p}} + \left\{mU(\mathbf{Y}_1-\mathbf{Y}_1^{\mathrm{T}})+\tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_1^{\mathrm{T}}-\mathbf{E}_1)+\tfrac{1}{2}m\frac{U}{l}(\mathbf{E}_2-\mathbf{E}_2^{\mathrm{T}})+m\frac{U}{l}\mathbf{H}\right\}\dot{\mathbf{p}}$$

$$+\left\{m\frac{U^2}{l}(\mathbf{Y}_2+\mathbf{Y}_2^{\mathrm{T}})-\tfrac{1}{2}m\frac{\dot{U}}{l}(\mathbf{E}_1+\mathbf{E}_2^{\mathrm{T}})+m\dot{U}\mathbf{Y}_1-m\frac{U^2}{l^2}\mathbf{F}\right.$$

$$\left.-mU^2\mathbf{Q}+EI_k\mathbf{M}+m(b\dot{\theta}^2-\dot{U})\mathbf{S}_1+m\dot{\theta}^2\mathbf{S}_2\right\}\mathbf{p}+mg\mathbf{\Phi}=0. \tag{50}$$

For case 1 with beam extensional motion shown in Fig. 2, the vibration is found to occur in the **i–k** plane due to the effect of gravity and extensional motion of the beam. There is no vibration in the **i–j** plane as the beam is assumed to be initially straight and stationary. With the prescribed rotational and extensional motion for case 2, vibrations are found to occur in both **i–j** and **i–k** planes. The tip displacements increase significantly towards the end of the motion due to the beam extension and rotation and the absence of damping, which tends to reduce the amplitude, as reported by Yuh and Young (1991). For cases 3 and 4 shown in Fig. 3, the frequency of oscillation is approximately proportional to l/l^2 for these motions due to the shortening length of the beam, consistent with the reported results by Tabarrok *et al.* (1974), and Yuh and Young (1991) without the effect of gravity. For relatively fast axial motions with the numerical results shown in Figs 4 and 5, there is less oscillatory motion for the duration of the prescribed motion, especially for shortening motion. For example, there are only two peaks for the curves shown in Fig. 5, compared with 12 peaks for the corresponding curves shown in Fig. 3. However, if one compares the first 0.2 s of the motions, the motion of a beam with relatively fast axial motion will appear to be more oscillatory. For these eight cases with setting angle equal to 90°, the rotational motion of the beam is found to have little effect on the vibration of the beam in the **i–k** plane as the two curves shown in Figs 2–5 for the vibration in the **i–k** plane are almost identical. The prescribed rotational motion is not fast enough to have significant effect on the vibration in the **i–k** plane.

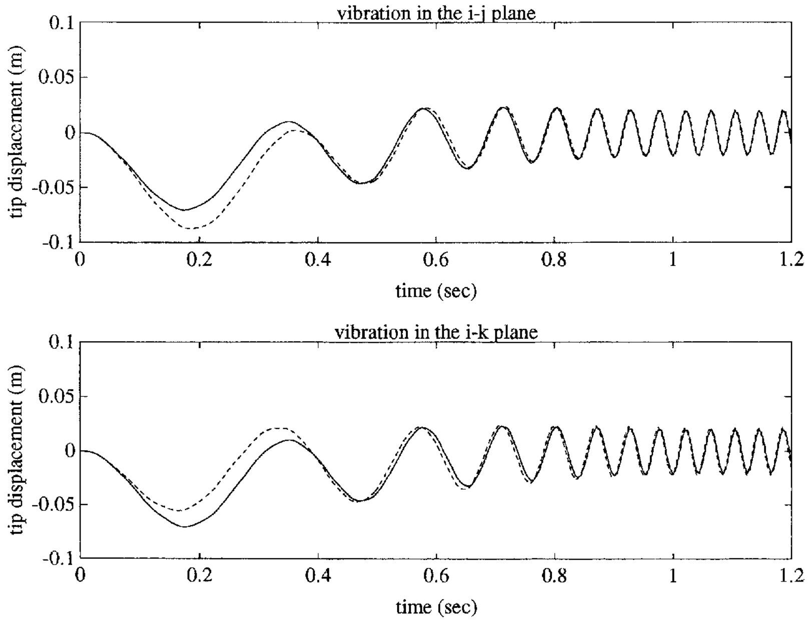

Fig. 6. Tip displacement for the beam with $\alpha = 45°$ for case 3 (——), case 4 (- - -).

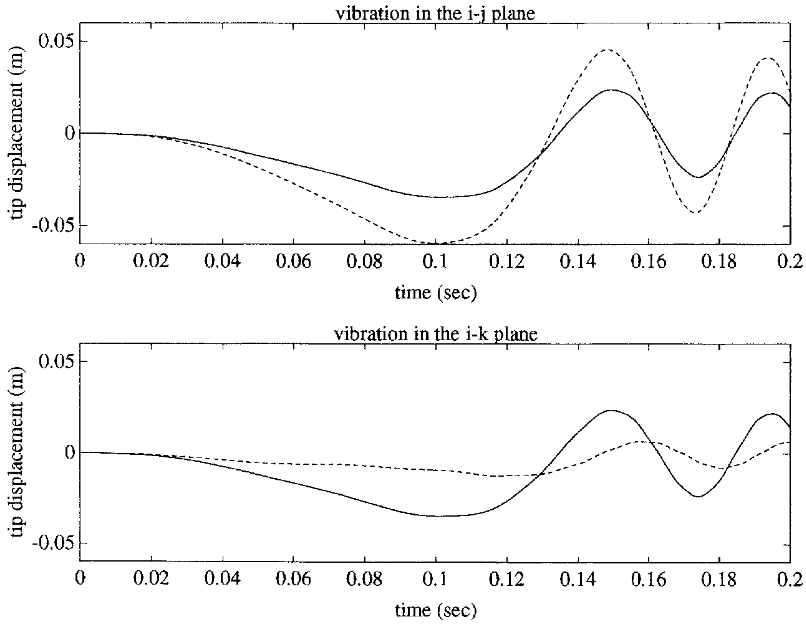

Fig. 7. Tip displacement for the beam with $\alpha = 45°$ for case 7 (——), case 8 (- - -).

The effect of setting angle is shown in Figs 6 and 7 for $\alpha = 45°$ and for the shortening motions described in cases 3, 4, 7 and 8. For this setting angle of $\alpha = 45°$, the two equations of motion presented in eqns (40) and (41) are no longer uncoupled. The behaviors of the beam in the **i**–**j** and **i**–**k** planes are identical when there is no prescribed rotational motion (see the solid curves in Figs 6 and 7), as the two equations of motion without the terms involving $\dot{\theta}$ and $\ddot{\theta}$ appear to be identical. The prescribed rotational motion, in addition to the axial motion of the beam, causes the behaviors in the two orthogonal planes to be different from each other. For example, for the case of fast axial motion shown in Fig. 7, the rotational motion causes the amplitude of vibration in the **i**–**k** plane to be smaller than that in the **i**–**j** plane, although the oscillatory patterns are similar for vibrations in both planes.

The behavior of the beam for other combinations of setting angles and prescribed axial and rotational motion can be easily computed using the two matrix equations (40) and (41) presented in this article. The effect of equivalent linear viscous damping can also be studied by incorporating damping terms such as $\mathbf{C}\dot{\mathbf{q}}$ and $\mathbf{C}\dot{\mathbf{p}}$ in the two matrix equations of motion where $\mathbf{C}$ is a matrix containing the damping factors.

4. CONCLUSION

A system of approximate equations of motion in matrix form is derived for the motion of an axially moving and rotating beam. A series of single-term functions is used for the assumed modes. The time derivatives of these shape functions are simpler in form than the time derivatives of beam functions used in the reported studies. The lateral deflections in the two orthogonal directions of the cross-section are coupled when the setting angle of the beam is not equal to 0° or 90°. Results of numerical simulations are presented for setting angle of 90° and 45° for various prescribed beam motions. For relatively slow motion of the beam, the tip displacement shows an oscillatory behavior for the duration of the prescribed motion. The variation of tip displacement is less oscillatory for relatively fast beam motion. However, if one compares the same duration of the initial motions, the motion of a beam with relatively fast axial motion will appear to be more oscillatory.

The present approximate equations of motion are formulated based on Euler beam theory. Therefore, the present analysis is not applicable for a beam with fast prescribed axial and rotational motions that result in large transverse deflections of the beam. A higher-order beam theory will need to be used under those circumstances.

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